

CHARACTERIZATION OF UNWANTED NOISE IN REALISTIC CAVITIES *

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Abstract

The problem of the description of absorption and scattering losses in high- Q cavities is studied. The considerations are based on quantum noise theories, hence the unwanted noise associated with scattering and absorption is taken into account by introduction of additional damping and noise terms in the quantum Langevin equations and input-output relations. Completeness conditions for the description of the cavity models obtained in this way are studied and corresponding replacement schemes are discussed.

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1 Introduction

Unwanted noise associated with absorption and scattering in high- Q cavities usually plays a crucial role in experiments in cavity quantum electro-

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dynamics (cavity QED) [1]. Even small values of the corresponding absorption/scattering coefficients may lead to dramatic changes of the quantum properties of the radiation. For typical high- Q cavities the unwanted losses can be of the same order of magnitude as the wanted, radiative losses due to the input–output coupling [2]. In such a case the process of quantum-state extraction from a high- Q cavity is characterized by efficiency of about 50%, [3]. This feature gives a serious restriction for the implementation of many proposals in cavity QED. Particularly, nowadays a lot of schemes for quantum-state engineering of the intracavity field are known. For example, in Ref. [4] a scheme for the generation of an arbitrary quantum state of the field is proposed. Also schemes for the generation of entangled states are known [5]. Unfortunately, due to the small efficiency of the quantum-state extraction, the states of the field may lose essential nonclassical properties after escaping from the cavity.

In the framework of quantum noise theories (QNT), a high- Q -cavity mode is usually considered as a harmonic oscillator interacting through the coupling mirror with a number of external modes. This leads to the description of the cavity mode in terms of a quantum Langevin equation and input–output relation [6]. The same result can be obtained in a quantum-field theoretical (QFT) approach [7]–[10] in appropriate limits.

For the description of unwanted losses of the intracavity field, it is sufficient to suppose the existence of a non-radiative input–output channel associated with absorption and scattering processes [3, 10]. However this model works properly, with respect to the output field, only in cases when the input ports of the cavity are unused. Indeed, it is clear that an input field can be absorbed or scattered in the coupling mirror. A detailed analysis shows that for a complete description of the unwanted noise one should take into account also the absorption/scattering losses inside the coupling mirrors [11].

As we will show below, the unwanted noise can be modeled by introducing blocks of beam splitters in an appropriately chosen replacement scheme, leading to additional noise terms in the quantum Langevin equation and the input–output relation. Such a description allows one to have a clear geometrical interpretation of the operators of unwanted noise as vectors in a unitary space. The minimum dimension of this space depends on the number of input–output ports. For example, for a one sided-cavity a two-dimensional space is sufficient, two-sided cavities require a three dimensional space and so on.

The requirement of preserving equal-time commutation rules leads to the conclusion that the c -number coefficients in the quantum Langevin equation and the input–output relations satisfy several constraints. In other words, their values belong to a certain multidimensional manifold. Therefore, the

problem of consistency and completeness of the corresponding replacement scheme can be solved by applying differential geometry [12].

In the present paper we formulate conditions of completeness for replacement schemes modeling unwanted noise in high- Q cavities and consider several examples. The paper is organized as follows. In Sec. 2 the mathematical model of unwanted noise in one-sided cavities is introduced. Examples of replacement schemes are considered in Sec. 3 and their applicability is discussed. Cavities with two and more input–output ports are considered in Sec. 4. A summary and some concluding remarks are given in Sec. 5.

2 Complete model of the unwanted noise

2.1 Idealized cavity model

Let us start our consideration by considering a one-sided cavity and reminding the standard QNT model, which does not include channels of unwanted noise [6, 7]. In this case the cavity-mode operator \hat{a}_{cav} obeys the quantum Langevin equation

$$\dot{\hat{a}}_{\text{cav}}(t) = -\left(i\omega_{\text{cav}} + \frac{\Gamma}{2}\right)\hat{a}_{\text{cav}}(t) + \mathcal{T}^{(\text{c})}\hat{b}_{\text{in}}(t), \quad (1)$$

where ω_{cav} is a resonant frequency of the cavity, Γ is the cavity decay rate, $\mathcal{T}^{(\text{c})}$ is the complex transmission coefficient describing injection of an input field into the cavity, and $\hat{b}_{\text{in}}(t)$ is the input-field operator satisfying the commutation rule

$$[\hat{b}_{\text{in}}(t_1), \hat{b}_{\text{in}}^\dagger(t_2)] = \delta(t_1 - t_2). \quad (2)$$

The output-field operator $\hat{b}_{\text{out}}(t)$ satisfying the commutation rule

$$[\hat{b}_{\text{out}}(t_1), \hat{b}_{\text{out}}^\dagger(t_2)] = \delta(t_1 - t_2) \quad (3)$$

is connected to the cavity-mode operator $\hat{a}_{\text{cav}}(t)$ and the input-field operator $\hat{b}_{\text{in}}(t)$ via the input–output relation

$$\hat{b}_{\text{out}}(t) = \mathcal{T}^{(\text{o})}\hat{a}_{\text{cav}}(t) + \mathcal{R}^{(\text{o})}\hat{b}_{\text{in}}(t). \quad (4)$$

Here $\mathcal{T}^{(\text{o})}$ is the complex transmission coefficient so that extraction of the cavity field becomes possible Ref. [3], and $\mathcal{R}^{(\text{o})}$ is the complex reflection coefficient at the cavity.

The solution of the quantum Langevin equation (1) can be written in the form

$$\hat{a}_{\text{cav}}(t) = \hat{a}_{\text{cav}}(0)e^{-(i\omega_{\text{cav}} + \Gamma/2)t} + \mathcal{T}^{(\text{c})}\int_0^t dt' e^{-(i\omega_{\text{cav}} + \Gamma/2)(t-t')}\hat{b}_{\text{in}}(t'). \quad (5)$$

Inserting it into the input–output relation (4), one obtains an expression for the output-field operator in terms of the input-field operator and the operator of the cavity mode at the initial time,

$$\begin{aligned}\hat{b}_{\text{out}}(t) = & \mathcal{T}^{(o)} \hat{a}_{\text{cav}}(0) e^{-(i\omega_{\text{cav}} + \Gamma/2)t} \\ & + \mathcal{T}^{(o)} \mathcal{T}^{(c)} \int_0^t dt' e^{-(i\omega_{\text{cav}} + \Gamma/2)(t-t')} \hat{b}_{\text{in}}(t') + \mathcal{R}^{(o)} \hat{b}_{\text{in}}(t).\end{aligned}\quad (6)$$

Assuming that the cavity-mode operator obeys the standard bosonic commutation relation at all times

$$[\hat{a}_{\text{cav}}(t), \hat{a}_{\text{cav}}^\dagger(t)] = 1, \quad (7)$$

and that the input-field operator commutes with the cavity-mode operator at the initial time $t = 0$, using Eqs. (2, 5, 7) one obtains the constraint

$$\Gamma = |\mathcal{T}^{(c)}|^2. \quad (8)$$

In a similar way from Eqs. (2, 3, 6, 7) we can see that the conditions

$$|\mathcal{R}^{(o)}|^2 = 1, \quad (9)$$

and

$$\mathcal{T}^{(o)} + \mathcal{T}^{(c)*} \mathcal{R}^{(o)} = 0 \quad (10)$$

are satisfied. A consequence of these constraints is the fact that the reflection coefficient $\mathcal{R}^{(o)}$ can be expressed in terms of $\mathcal{T}^{(o)}$ and $\mathcal{T}^{(c)}$ as

$$\mathcal{R}^{(o)} = -\frac{\mathcal{T}^{(o)}}{\mathcal{T}^{(c)*}}. \quad (11)$$

In summary, the QNT model of a cavity without unwanted noise includes the quantum Langevin equation (1), the input–output relation (4) and the constraints for the c -number coefficients (8-10). In particular, the constraint (8) describes the relation between the cavity decay rate and the coefficient $\mathcal{T}^{(c)}$. A consequence of the constraints is the equality of the absolute values of the transmission coefficients $\mathcal{T}^{(o)}$ and $\mathcal{T}^{(c)}$.

2.2 Realistic cavity model

As it has already been mentioned in the introduction, unwanted noise can be included in the QNT model through introduction of additional noise terms.

Indeed, for a description of absorption and scattering of the cavity field, one can consider the quantum Langevin equation

$$\dot{\hat{a}}_{\text{cav}} = - \left[i\omega_{\text{cav}} + \frac{1}{2}\Gamma \right] \hat{a}_{\text{cav}} + \mathcal{T}^{(c)} \hat{b}_{\text{in}}(t) + \hat{C}^{(c)}(t). \quad (12)$$

Accordingly, the possibility of absorption and scattering of the input field is described by introduction of an additional term into the input-output relation

$$\hat{b}_{\text{out}}(t) = \mathcal{T}^{(o)} \hat{a}_{\text{cav}}(t) + \mathcal{R}^{(o)} \hat{b}_{\text{in}}(t) + \hat{C}^{(o)}(t). \quad (13)$$

In these equations, the operators of unwanted noise, $\hat{C}^{(c)}(t)$ and $\hat{C}^{(o)}(t)$, which commute with the input-field operator $\hat{b}_{\text{in}}(t)$, the cavity-field operator at the initial time $t = 0$, $\hat{a}_{\text{cav}}(0)$, satisfy the following commutation rules:

$$[\hat{C}^{(c)}(t_1), \hat{C}^{(c)\dagger}(t_2)] = |\mathcal{A}^{(c)}|^2 \delta(t_1 - t_2), \quad (14)$$

$$[\hat{C}^{(o)}(t_1), \hat{C}^{(o)\dagger}(t_2)] = |\mathcal{A}^{(o)}|^2 \delta(t_1 - t_2), \quad (15)$$

$$[\hat{C}^{(c)}(t_1), \hat{C}^{(o)\dagger}(t_2)] = \Xi \delta(t_1 - t_2). \quad (16)$$

Here, $\mathcal{A}^{(c)}$, $\mathcal{A}^{(o)}$, and Ξ , are c -number absorption/scattering coefficients. The set of the coefficients Γ , ω_{cav} , $\mathcal{T}^{(c)}$, $\mathcal{T}^{(o)}$, $\mathcal{R}^{(o)}$, $|\mathcal{A}^{(c)}|^2$, $|\mathcal{A}^{(o)}|^2$, and Ξ characterizes the cavity with unwanted noise.

Similar to the case of an idealized cavity, one can show that the requirement of preserving the commutation rules leads to the following constraints:

$$\Gamma = |\mathcal{A}^{(c)}|^2 + |\mathcal{T}^{(c)}|^2, \quad (17)$$

$$|\mathcal{R}^{(o)}|^2 + |\mathcal{A}^{(o)}|^2 = 1, \quad (18)$$

$$\mathcal{T}^{(o)} + \mathcal{T}^{(c)*} \mathcal{R}^{(o)} + \Xi = 0. \quad (19)$$

Hence, we conclude that the c -number coefficients describing a realistic cavity should belong to the manifold defined by Eqs. (17-19). As it is well known from differential geometry, each manifold can be described by means of independent parameters [12]. Particularly, this means that the c -number coefficients can be expressed in terms of appropriately chosen parameters. The corresponding parametrization should cover the whole manifold. In other words, it should describe all possible cavities. Otherwise, one gets a degenerate model, which describes just a restricted class of the cavities.

The easiest way for the parametrization of the considered manifold follows directly from Eqs. (17-19). Indeed, the coefficients Γ , ω_{cav} , $\mathcal{T}^{(c)}$, $\mathcal{T}^{(o)}$, $\mathcal{R}^{(o)}$ can

be considered as independent parameters, and the coefficients $|\mathcal{A}^{(c)}|^2$, $|\mathcal{A}^{(o)}|^2$, Ξ are expressed in terms of them. Such a parametrization can be convenient in experimental investigations, because the corresponding parameters have a clear physical interpretation. However, in the theoretical investigation one should use this parametrization very carefully. In fact, not all values of these independent parameters belong to the manifold. Particularly, their values are necessarily restricted by the inequalities

$$|\mathcal{T}^{(c)}|^2 \leq \Gamma, \quad (20)$$

$$|\mathcal{T}^{(o)}|^2 \leq \Gamma, \quad (21)$$

$$|\mathcal{R}^{(o)}|^2 \leq 1, \quad (22)$$

$$|\mathcal{T}^{(o)} + \mathcal{T}^{(c)*}\mathcal{R}^{(o)}|^2 \leq \left(\Gamma - |\mathcal{T}^{(o)}|^2\right) \left(1 - |\mathcal{R}^{(o)}|^2\right), \quad (23)$$

which are derived from Eqs. (17-19). The fact that the parameters satisfy these inequalities is not a sufficient condition for describing realistic cavities. However, if they do not satisfy these conditions, one can conclude that the chosen values are not physically consistent ones.

2.3 Operators of unwanted noise

It is worth noting that the operators of unwanted noise, $\hat{C}^{(c)}(t_1)$ and $\hat{C}^{(o)}(t_1)$, can be considered as two vectors in a unitary vector space. In particular, the scalar product of two arbitrary vectors $\hat{C}^{(a)}(t)$ and $\hat{C}^{(b)}(t)$ in this space can be defined by

$$(\hat{C}^{(a)}, \hat{C}^{(b)}) = \int dt_1 [\hat{C}^{(a)\dagger}(t_1), \hat{C}^{(b)}(t_2)]. \quad (24)$$

In this interpretation, $|\mathcal{A}^{(c)}|$ and $|\mathcal{A}^{(o)}|$ can be considered as the absolute values of the vectors $\hat{C}^{(c)}(t)$ and $\hat{C}^{(o)}(t)$, respectively, whereas Ξ defines the (complex) angle between them, i.e.,

$$\Xi = |\mathcal{A}^{(c)}| |\mathcal{A}^{(o)}| e^{i\kappa} \cos \zeta, \quad (25)$$

where κ and ζ are real.

The vectors $\hat{C}^{(c)}(t)$ and $\hat{C}^{(o)}(t)$ can be expanded in an orthogonal basis,

$$\hat{C}^{(c)}(t) = \sum_k \mathcal{A}_{(k)}^{(c)} \hat{c}_{\text{in}}^{(k)}(t), \quad (26)$$

$$\hat{C}^{(o)}(t) = \sum_k \mathcal{A}_{(k)}^{(o)} \hat{c}_{\text{in}}^{(k)}(t), \quad (27)$$

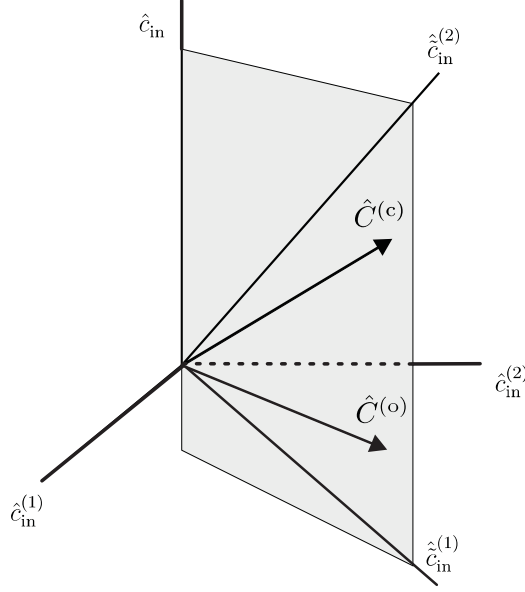


Figure 1: Geometrical interpretation of the operators of the unwanted noise. The operators $\hat{c}_{\text{in}}^{(1)}(t)$, $\hat{c}_{\text{in}}^{(2)}(t)$ and $\hat{c}_{\text{in}}(t)$ correspond to a three-dimensional representation of the unwanted noise. The operators $\hat{\tilde{c}}_{\text{in}}^{(1)}(t)$ and $\hat{\tilde{c}}_{\text{in}}^{(2)}(t)$ correspond to an equivalent representation in a two-dimensional space.

which implies that different representations of the operators of the unwanted noise can be obtained. It is clear (in full analogy with usual geometry) that two vectors always belong to a two-dimensional plane, see Fig. 1. This plane may be considered as a two-dimensional unitary vector space, with the two basis vectors $\hat{\tilde{c}}_{\text{in}}^{(1)}(t)$ and $\hat{\tilde{c}}_{\text{in}}^{(2)}(t)$ playing the role of appropriately chosen operators of the unwanted noise. Consequently, it is sufficient to have two basis operators for a complete description of the unwanted noise of a (one-sided) cavity. However, as we will show in the following, in some cases it is convenient to use representations with a larger number of dimensions, with particular emphasis on the three-dimensional case.

3 Replacement schemes

3.1 Complete scheme

The method of replacement schemes, widely used in quantum optics, can be applied to the formulation of a parametrization which works with all the

values of independent parameters. The main idea is to simulate the channels of unwanted noise by an additional input–output port and a block of beam splitters as illustrated in Fig. 2. The additional input–output port models the losses typically responsible for the absorption and scattering of the radiation, which escapes from the cavity, whereas the block of beam splitters describes losses like absorption and scattering of the input field entering the cavity. In Fig. 2, the symmetrical beam-splitters BS_1 and BS_2 simulate the unwanted noise inside the coupling mirror. Moreover, due to the requirements of completeness one must include in the scheme the asymmetrical $U(2)$ beam-splitter BS_3 , which simulates feedback.

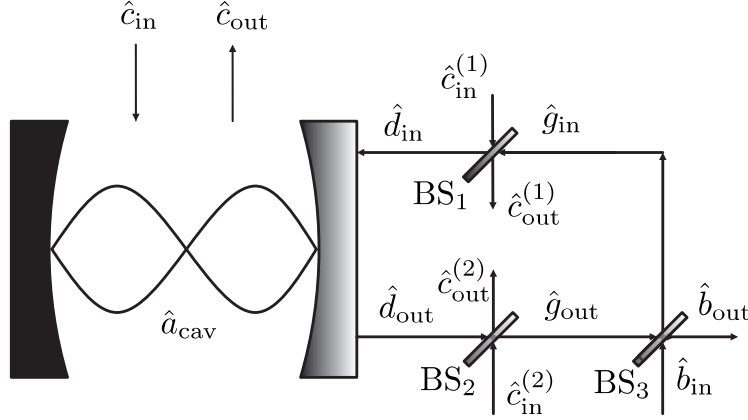


Figure 2: Replacement scheme for the simulation of unwanted noise in a high- Q cavity. The $SU(2)$ beam splitters BS_1 and BS_2 model unwanted noise inside the coupling mirror and $U(2)$ beam splitter BS_3 simulates feedback.

Using the quantum Langevin equation and the input–output relation for a cavity with two input–output ports as well as the input–output relations for each beam splitter separately, we obtain Eqs. (4, 12), where the operators $\hat{C}^{(c)}(t)$ and $\hat{C}^{(o)}(t)$ have the form

$$\hat{C}^{(c)}(t) = \mathcal{A}_{(1)}^{(c)} \hat{c}_{\text{in}}^{(1)}(t) + \mathcal{A}_{(2)}^{(c)} \hat{c}_{\text{in}}^{(2)}(t) + \mathcal{A} \hat{c}_{\text{in}}(t), \quad (28)$$

$$\hat{C}^{(o)}(t) = \mathcal{A}_{(1)}^{(o)} \hat{c}_{\text{in}}^{(1)}(t) + \mathcal{A}_{(2)}^{(o)} \hat{c}_{\text{in}}^{(2)}(t). \quad (29)$$

The c -number coefficients in the quantum Langevin equation and the input–output relation are expressed in terms of the beam-splitter transmission and reflection coefficients $\mathcal{T}^{(k)}$ and $\mathcal{R}^{(k)}$ respectively ($k = 1, 2, 3$), the phase factor $\varphi^{(3)}$ of the beam splitter BS_3 , the resonance frequency ω_0 , the radiation and

absorption decay rates of the “primary” cavity in the scheme, γ and $|\mathcal{A}|^2$ respectively, as follows:

$$\Gamma = \gamma \frac{1 - |\mathcal{R}^{(3)}|^2 |\mathcal{T}^{(1)}|^2 |\mathcal{T}^{(2)}|^2}{|1 - \mathcal{R}^{(3)*} \mathcal{T}^{(1)} \mathcal{T}^{(2)}|^2} + |\mathcal{A}|^2 \quad (30)$$

$$\omega_{\text{cav}} = \omega_0 - i \frac{\gamma}{2} \frac{\mathcal{R}^{(3)*} \mathcal{T}^{(1)} \mathcal{T}^{(2)} - \mathcal{R}^{(3)} \mathcal{T}^{(1)*} \mathcal{T}^{(2)*}}{|1 - \mathcal{R}^{(3)*} \mathcal{T}^{(1)} \mathcal{T}^{(2)}|^2} \quad (31)$$

$$\mathcal{T}^{(c)} = \sqrt{\gamma} \frac{\mathcal{T}^{(1)} \mathcal{T}^{(3)*}}{1 - \mathcal{R}^{(3)*} \mathcal{T}^{(1)} \mathcal{T}^{(2)}} \quad (32)$$

$$\mathcal{A}_{(1)}^{(c)} = \sqrt{\gamma} \frac{\mathcal{R}^{(1)}}{1 - \mathcal{R}^{(3)*} \mathcal{T}^{(1)} \mathcal{T}^{(2)}}, \quad (33)$$

$$\mathcal{A}_{(2)}^{(c)} = -\sqrt{\gamma} \frac{\mathcal{T}^{(1)} \mathcal{R}^{(2)} \mathcal{R}^{(3)*}}{1 - \mathcal{R}^{(3)*} \mathcal{T}^{(1)} \mathcal{T}^{(2)}}, \quad (34)$$

$$\mathcal{T}^{(o)} = \sqrt{\gamma} e^{i\varphi^{(3)}} \frac{\mathcal{T}^{(2)} \mathcal{T}^{(3)}}{1 - \mathcal{R}^{(3)*} \mathcal{T}^{(1)} \mathcal{T}^{(2)}}, \quad (35)$$

$$\mathcal{R}^{(o)} = e^{i\varphi^{(3)}} \frac{\mathcal{R}^{(3)} - \mathcal{T}^{(1)} \mathcal{T}^{(2)}}{1 - \mathcal{R}^{(3)*} \mathcal{T}^{(1)} \mathcal{T}^{(2)}}, \quad (36)$$

$$\mathcal{A}_{(1)}^{(o)} = -e^{i\varphi^{(3)}} \frac{\mathcal{T}^{(2)} \mathcal{R}^{(1)} \mathcal{T}^{(3)}}{1 - \mathcal{R}^{(3)*} \mathcal{T}^{(1)} \mathcal{T}^{(2)}}, \quad (37)$$

$$\mathcal{A}_{(2)}^{(o)} = e^{i\varphi^{(3)}} \frac{\mathcal{R}^{(2)} \mathcal{T}^{(3)}}{1 - \mathcal{R}^{(3)*} \mathcal{T}^{(1)} \mathcal{T}^{(2)}}. \quad (38)$$

Each complex coefficient $\mathcal{T}^{(k)}$ and $\mathcal{R}^{(k)}$ can be expressed in terms of three real independent parameters $\theta^{(k)}$, $\mu^{(k)}$ and $\nu^{(k)}$:

$$\mathcal{T}^{(k)} = \cos \theta^{(k)} e^{i\mu^{(k)}}, \quad (39)$$

$$\mathcal{R}^{(k)} = \sin \theta^{(k)} e^{i\nu^{(k)}}. \quad (40)$$

Therefore, one gets the parametrization of the manifold by means of the independent parameters.

In order to check the completeness of the proposed parametrization, one should first present Eqs. (30-38) in the form of real functions of real parameters. Next, one should build the matrix containing the first derivatives of these functions. The determinant of this function should be non-zero. The corresponding calculations have been performed by using Mathematica 5.1. It has been found that the parametrization corresponding to the considered replacement scheme and given by Eqs. (30-38) completely describes (one-sided) cavities with unwanted noise.

3.2 Degenerate schemes

Let us consider examples of replacement schemes, referred to as degenerate replacement schemes, which do not describe all possible cavities. As a rule, degenerate schemes can be obtained from the complete scheme by removing one or more elements. Cavities modeled by degenerate replacement schemes usually obey some constraints in addition to Eqs. (17-19). In other words, such cavities correspond to points in a certain sub-manifold rather than in the whole manifold.

The first example is the class of cavities obtained from Fig. 2 by removing the beam splitters BS_1 and BS_2 . For such cavities, the input field does not suffer from losses when it enters the cavity. The operators $\hat{C}^{(c)}(t)$ and $\hat{C}^{(o)}(t)$ in the quantum Langevin equation and the input–output relation in this case read as

$$\hat{C}^{(c)}(t) = \mathcal{A} \hat{c}_{\text{in}}(t), \quad (41)$$

$$\hat{C}^{(o)}(t) = 0. \quad (42)$$

It is clear that for such a cavity the noise term associated with absorption and scattering is only included in the quantum Langevin equation – a model, which can be used for special applications [3, 10]. The corresponding parametrization is a rather trivial one, the additional constraint has the form

$$|\mathcal{R}^{(o)}|^2 = 1. \quad (43)$$

A consequence of this fact is that $\mathcal{A}^{(o)} = 0$. Moreover, such a cavity has some properties very close to the idealized cavity without channels of unwanted noise. The transmission coefficients $\mathcal{T}^{(o)}$ and $\mathcal{T}^{(c)}$ are equal and Eq. (11) holds true for the reflection coefficient $\mathcal{R}^{(o)}$.

Another example of a degenerate scheme can be obtained from the complete scheme in Fig. 2 by removing the (non-radiative) input–output channels \hat{c}_{in} , \hat{c}_{out} and the beam splitter BS_3 associated with the feedback. In this case, the operators $\hat{C}^{(c)}(t)$ and $\hat{C}^{(o)}(t)$ have the form

$$\hat{C}^{(c)}(t) = \mathcal{A}_{(1)}^{(c)} \hat{c}_{\text{in}}^{(1)}(t), \quad (44)$$

$$\hat{C}^{(o)}(t) = \mathcal{A}_{(1)}^{(o)} \hat{c}_{\text{in}}^{(1)}(t) + \mathcal{A}_{(2)}^{(o)} \hat{c}_{\text{in}}^{(2)}(t). \quad (45)$$

Although both the quantum Langevin equation and the input–output relation contain noise terms associated with unwanted losses, the scheme is not a complete one. The corresponding parametrization can be written in the form

$$\Gamma = \gamma, \quad (46)$$

$$\omega_{\text{cav}} = \omega_0, \quad (47)$$

$$\mathcal{T}^{(c)} = \sqrt{\gamma} \mathcal{T}^{(1)}, \quad (48)$$

$$\mathcal{A}_{(1)}^{(c)} = \sqrt{\gamma} \mathcal{R}^{(1)}, \quad (49)$$

$$\mathcal{T}^{(o)} = \sqrt{\gamma} \mathcal{T}^{(2)}, \quad (50)$$

$$\mathcal{R}^{(o)} = -\mathcal{T}^{(1)} \mathcal{T}^{(2)}, \quad (51)$$

$$\mathcal{A}_{(1)}^{(o)} = -\mathcal{R}^{(1)} \mathcal{T}^{(2)}, \quad (52)$$

$$\mathcal{A}_{(2)}^{(o)} = \mathcal{R}^{(2)}. \quad (53)$$

One can easily prove by direct calculations that along with Eqs. (17-19) this parametrization satisfy the following additional constraint

$$\frac{\mathcal{T}^{(o)} \mathcal{T}^{(c)}}{\Gamma} + \mathcal{R}^{(o)} = 0. \quad (54)$$

It is worth noting that the physics behind this degenerate scheme is closely related to that of a cavity without unwanted noise. Indeed, the unwanted noise can be regarded as noise associated with the transmission channel. Hence, the losses modeled in this way cannot affect the decay rate of the intracavity field, but some properties of the external field are changed. Particularly, for such cavities it is impossible to combine a cavity mode and an input mode in an output mode.

4 Two-sided cavities

So far we have considered one-sided cavities. In various physical applications, e.g., the generation of squeezed light in optical parametric amplification [13], it is necessary to consider the problem of unwanted noise in cavities with two (or even more) radiative input–output ports. Let us generalize the replacement-scheme method developed above to a two-sided cavity, as is sketched in Fig. 3. It is straightforward to show that the generalization of Eqs. (12) and (13) is

$$\begin{aligned} \dot{\hat{a}}_{\text{cav}} = & - \left[i\omega_{\text{cav}} + \frac{1}{2}\Gamma \right] \hat{a}_{\text{cav}} + \mathcal{T}_{(R)}^{(c)} \hat{b}_{\text{in}}^{(R)}(t) + \mathcal{T}_{(L)}^{(c)} \hat{b}_{\text{in}}^{(L)}(t) \\ & + \mathcal{A}_{(1)}^{(c)} \hat{c}_{\text{in}}^{(1)}(t) + \mathcal{A}_{(2)}^{(c)} \hat{c}_{\text{in}}^{(2)}(t) + \mathcal{A}_{(3)}^{(c)} \hat{c}_{\text{in}}^{(3)}(t) + \mathcal{A}_{(4)}^{(c)} \hat{c}_{\text{in}}^{(4)}(t) + \mathcal{A} \hat{c}_{\text{in}}(t), \end{aligned} \quad (55)$$

$$\begin{aligned} \hat{b}_{\text{out}}^{(R)}(t) = & \mathcal{T}_{(R)}^{(o)} \hat{a}_{\text{cav}}(t) + \mathcal{R}_{(R)}^{(o)} \hat{b}_{\text{in}}^{(R)}(t) \\ & + \mathcal{A}_{(1)}^{(o)} \hat{c}_{\text{in}}^{(1)}(t) + \mathcal{A}_{(2)}^{(o)} \hat{c}_{\text{in}}^{(2)}(t), \end{aligned} \quad (56)$$

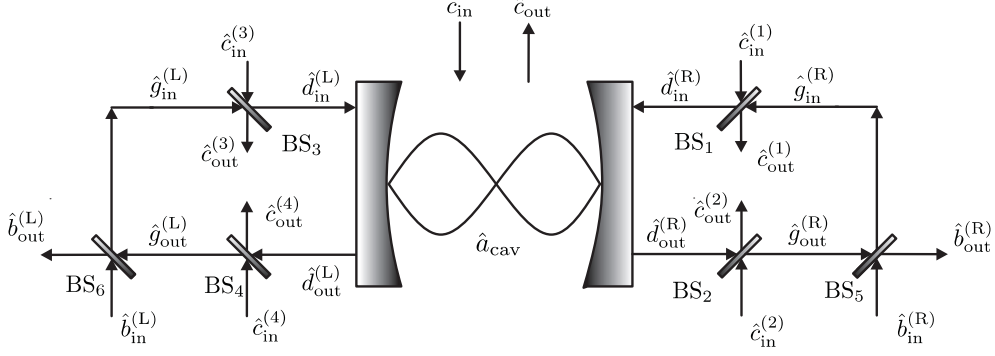


Figure 3: Replacement scheme for modeling unwanted noise in a two-sided cavity. The symmetrical $SU(2)$ -type beam splitters BS₁, BS₂, BS₃, and BS₄ model the unwanted noise in the two coupling mirrors, and the asymmetrical $U(2)$ -type beam splitters BS₅ and BS₆ simulate some feedback.

$$\begin{aligned} \hat{b}_{\text{out}}^{(L)}(t) &= \mathcal{T}_{(L)}^{(o)} \hat{a}_{\text{cav}}(t) + \mathcal{R}_{(L)}^{(o)} \hat{b}_{\text{in}}^{(L)}(t) \\ &\quad + \mathcal{A}_{(3)}^{(o)} \hat{c}_{\text{in}}^{(3)}(t) + \mathcal{A}_{(4)}^{(o)} \hat{c}_{\text{in}}^{(4)}(t), \end{aligned} \quad (57)$$

where the coefficients in these equations can be obtained in a similar manner as for the case of a one-sided cavity.

As in the case of a one-sided cavity, the c -number coefficients in Eqs. (55)–(57) are also not independent of each other. From considering the commutation relation for the cavity-mode operator, one obtains

$$\begin{aligned} \Gamma &= |\mathcal{A}|^2 + |\mathcal{A}_{(1)}^{(c)}|^2 + |\mathcal{A}_{(2)}^{(c)}|^2 + |\mathcal{A}_{(3)}^{(c)}|^2 + |\mathcal{A}_{(4)}^{(c)}|^2 \\ &\quad + |\mathcal{T}_{(R)}^{(c)}|^2 + |\mathcal{T}_{(L)}^{(c)}|^2. \end{aligned} \quad (58)$$

With regard to the right-hand wall of the cavity, the requirement of preserving the commutation rules implies that

$$|\mathcal{R}_{(R)}^{(o)}|^2 + |\mathcal{A}_{(1)}^{(o)}|^2 + |\mathcal{A}_{(2)}^{(o)}|^2 = 1, \quad (59)$$

$$\mathcal{T}_{(R)}^{(o)} + \mathcal{T}_{(R)}^{(c)*} \mathcal{R}_{(R)}^{(o)} + \mathcal{A}_{(1)}^{(c)*} \mathcal{A}_{(1)}^{(o)} + \mathcal{A}_{(2)}^{(c)*} \mathcal{A}_{(2)}^{(o)} = 0. \quad (60)$$

Finally, for the field outgoing from the left-hand wall of the cavity, one can show that

$$|\mathcal{R}_{(L)}^{(o)}|^2 + |\mathcal{A}_{(3)}^{(o)}|^2 + |\mathcal{A}_{(4)}^{(o)}|^2 = 1, \quad (61)$$

$$\mathcal{T}_{(L)}^{(o)} + \mathcal{T}_{(L)}^{(c)*} \mathcal{R}_{(L)}^{(o)} + \mathcal{A}_{(3)}^{(c)*} \mathcal{A}_{(3)}^{(o)} + \mathcal{A}_{(4)}^{(c)*} \mathcal{A}_{(4)}^{(o)} = 0. \quad (62)$$

It should be pointed out that the operators of unwanted noise – represented by $\hat{c}_{\text{in}}^{(1)}$, $\hat{c}_{\text{in}}^{(2)}$, $\hat{c}_{\text{in}}^{(3)}$ and $\hat{c}_{\text{in}}^{(4)}$ in Eqs. (55-57) – can be also represented in

other forms. Since the corresponding expressions in Eqs. (55)–(57) contain only three linear combinations of the operators \hat{c}_{in} , $\hat{c}_{\text{in}}^{(1)}$, $\hat{c}_{\text{in}}^{(2)}$, $\hat{c}_{\text{in}}^{(3)}$ and $\hat{c}_{\text{in}}^{(4)}$, one can conclude that there exist equivalent formulations of these equations with three independent operators of unwanted noise.

5 Summary and conclusions

The concept of replacement schemes is a very helpful tool to study, within the framework of QNT, the effect of unwanted noise associated with absorption and scattering in realistic high- Q cavities, which leads to the appearance of additional noise terms in both the standard quantum Langevin equations and the standard input–output relations attributed to them.

An important mathematical feature is the fact that the c -number coefficients in the quantum Langevin equations and in the input–output relation are not independent ones. In particular, the requirement of preserving typical commutation rules leads to the appearance of several constraints. Hence, the corresponding values of the coefficients can be regarded as belonging to a certain manifold.

A consistent physical description of realistic cavities requires to formulate the theory in terms of independent parameters only. In other words, one must consider the parametrization of the manifold. So, one can formally express some of the c -number coefficients in terms of the other ones and simply consider the latter as independent parameters. Another, more physical way is a parametrization on the basis of appropriately chosen replacement schemes.

The method of replacement schemes in fact allows one to distinguish, with respect to the unwanted noise, between qualitatively different cavity models. Roughly speaking, one can distinguish between non-degenerate and degenerate replacement schemes. In contrast to non-degenerate schemes, where the parametrization completely describes cavities with unwanted losses, degenerate schemes do not describe all possible cavities but only special classes.

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